PROOF OF THE FLOHR–GRABOW–KOEHN CONJECTURES FOR CHARACTERS OF LOGARITHMIC CONFORMAL FIELD THEORY

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Abstract. In a recent paper Flohr, Grabow and Koehn conjectured that the characters of the logarithmic conformal field theory $c_{k,1}$, of central charge $c=1-6(k-1)^2/k$, admit fermionic representations labelled by the Lie algebra D_k. In this note we provide a simple analytic proof of this conjecture.

1. Introduction

In the past two decades an almost complete understanding of the analytic and combinatorial structure of fermionic character representations for the minimal unitary models $M(p,p')$ in conformal field theory has been obtained [3–6, 10–13, 16, 19–22]. In a recent paper Flohr, Grabow and Koehn (FGK) [9] took the first tentative steps towards extending these results to the realm of logarithmic conformal field theory (LCFT). FGK focused on one of the simplest examples of a LCFT; the $\mathcal{W}(2,2k-1,2k-1,2k-1)$ series of triplet algebras of central charge

$$c=1-\frac{6(k-1)^2}{k},$$

denoted as the $c_{k,1}$ models for short. Surprisingly, FGK conjectured that the characters of the $c_{k,1}$ models may be described in terms of the Lie algebra D_k. For example, if B denotes the inverse Cartan matrix of D_k, then, conjecturally,

$$\chi_\lambda(\tau)=q^{\varphi_\lambda}\sum_{\substack{n_1,\dots,n_k=0\\ n_{k-1}\equiv n_k\ (2)}}^{\infty}\frac{q^{\sum_{i,j=1}^k B_{ij}n_in_j+\frac{\lambda}{2}(n_{k-1}-n_k)}}{(q;q)_{n_1}\cdots(q;q)_{n_k}}. \tag{1}$$

Here χ_λ for $\lambda\in\{0,1,\dots,k\}$ is a $c_{k,1}$ character, $\varphi_\lambda=\lambda^2/(4k)-1/24$, $q=\exp(2\pi\mathrm{i}\tau)$, $(q;q)_n=(1-q)(1-q^2)\cdots(1-q^n)$, and $a\equiv b\ (c)$ is shorthand for $a\equiv b \pmod c$.

In support of (1) FGK provided a proof for the degenerate case $k=2$ (in which case B is half the 2×2 identity matrix), and showed that in the $q\to 1^-$ limit (1) gives rise to the well-known D_k dilogarithm identity

$$2L\Big(\frac{1}{k}\Big)+\sum_{j=2}^{k-1}L\Big(\frac{1}{j^2}\Big)=\frac{\pi^2}{6},$$

where $L(x)$ is the Rogers dilogarithm [15]. (The D_k nature of the above identity lies with the fact that $x=(1/4,1/9,\dots,1/(k-1)^2,1/k,1/k)$ solves the simultaneous equations $x_i=\prod_{j=1}^k(1-x_j)^{2B_{ij}}$ for $1\le i\le k$.)

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In this paper we provide an analytic proof of (1) and its allied $c_{k,1}$ character identities. Key is the observation that the matrix B contains the submatrix

$$T = (\min(i,j))_{1\le i,j\le k-2}$$

which itself admits character identities similar to (1). Indeed it is a classical result — first discovered by Andrews [1] in the context of partition theory — that

$$\chi^{\mathrm{Vir}}_{\lambda}(\tau) = q^{\phi_\lambda} \sum_{n_1,\dots,n_{k-2}=0}^{\infty} \frac{q^{\sum_{i,j=1}^{k-2} T_{ij}n_in_j + \sum_{i=\lambda}^{k-2}(i-\lambda+1)n_i}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}}, \tag{2}$$

where $\chi^{\mathrm{Vir}}_{\lambda}$ for $\lambda\in\{1,\dots,k-1\}$ are the characters of the Virasoro minimal models $M(2,2k-1)$ and

$$\phi_\lambda = \frac{(2\lambda-2k+1)^2}{8(2k-1)} - \frac{1}{24}.$$

It is the connection between the conjectural (1) and Andrews' (2) that will play a crucial role in our proof.

2. $c_{k,1}$ character formulas

We will not formally define the characters of the $c_{k,1}$ models but simply state their bosonic representation as obtained in [8].

Throughout we let $\tau\in\mathbb{C}$, $\mathrm{Im}(\tau)>0$ and $q\in\mathbb{C}$ be related by $q=\exp(2\pi\mathrm{i}\tau)$, so that $|q|<1$. This will make all q-series considered in this paper absolutely convergent so that we need not concern ourselves with order of summation in multiple series. We also use the standard q-notations

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \qquad\text{and}\qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

and Dedekind's eta-function

$$\eta(\tau) = q^{1/24}(q;q)_\infty.$$

Finally we need the theta and affine theta functions

$$\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}+\frac{\lambda}{2k}} q^{kn^2},$$

$$(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}+\frac{\lambda}{2k}} 2kn\, q^{kn^2}.$$

With the above notation the following bosonic character formulas corresponding to the $c_{k,1}$ LCFT hold [8]:

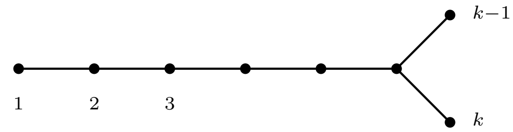
$$\chi_\lambda(\tau) = \frac{\Theta_{\lambda,k}(\tau)}{\eta(\tau)},$$

$$\chi^+_\lambda(\tau) = \frac{(k-\lambda)\,\Theta_{\lambda,k}(\tau) + (\partial\Theta)_{\lambda,k}(\tau)}{k\,\eta(\tau)} = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}(2n+1)q^{k(n+\frac{\lambda}{2k})^2},$$

$$\chi^-_\lambda(\tau) = \frac{\lambda\,\Theta_{\lambda,k}(\tau) - (\partial\Theta)_{\lambda,k}(\tau)}{k\,\eta(\tau)} = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}}2n\, q^{k(n-\frac{\lambda}{2k})^2},$$

with $\lambda\in\{0,1,\dots,k\}$ in χ_λ and $\lambda\in\{1,\dots,k-1\}$ in $\chi^{\pm}_\lambda$.

To state the fermionic expressions of FGK we let B be the inverse Cartan matrix of the Lie algebra D_k with labelling of the vertices of the Dynkin diagram given by



Hence

$$B_{k,k-1} = B_{k-1,k} = \frac{k-2}{4}, \qquad B_{k-1,k-1} = B_{k,k} = \frac{k}{4}, \tag{3a}$$

$$B_{i,k-1} = B_{i,k} = B_{k-1,i} = B_{k,i} = \frac{i}{2} \qquad 1 \le i \le k-2, \tag{3b}$$

$$B_{ij} = \min(i,j) \qquad 1 \le i, j, \le k-2. \tag{3c}$$

Then the conjectures of FGK correspond to

$$\chi_\lambda(\tau) = q^{\varphi_\lambda} \sum_{\substack{n_1,\dots,n_k=0 \\ n_{k-1}\equiv n_k \ (2)}}^{\infty} \frac{q^{\sum_{i,j=1}^k B_{ij}n_in_j + \frac{\lambda}{2}(n_{k-1}-n_k)}}{(q;q)_{n_1}\cdots(q;q)_{n_k}},$$

$$\chi_\lambda^+(\tau) = q^{\varphi_\lambda} \sum_{\substack{n_1,\dots,n_k=0 \\ n_{k-1}\equiv n_k \ (2)}}^{\infty} \frac{q^{\sum_{i,j=1}^k B_{ij}n_in_j + \sum_{i=k-\lambda}^{k-2}(i-k+\lambda+1)n_i + \frac{\lambda}{2}(n_{k-1}+n_k)}}{(q;q)_{n_1}\cdots(q;q)_{n_k}}$$

and

$$\chi_{k-\lambda}^-(\tau) = q^{\varphi_\lambda} \sum_{\substack{n_1,\dots,n_k=0 \\ n_{k-1}\not\equiv n_k \ (2)}}^{\infty} \frac{q^{\sum_{i,j=1}^k B_{ij}n_in_j + \sum_{i=k-\lambda}^{k-2}(i-k+\lambda+1)n_i + \frac{\lambda}{2}(n_{k-1}+n_k)}}{(q;q)_{n_1}\cdots(q;q)_{n_k}},$$

where

$$\varphi_\lambda = \frac{\lambda^2}{4k} - \frac{1}{24}.$$

Note that the last two expressions have identical summand and differ only in the restriction on the parity of $n_{k-1}+n_k$.

In addition to the above three conjectures we will also prove that

$$\chi_{k-\lambda}(\tau) = q^{\varphi_\lambda} \sum_{\substack{n_1,\dots,n_k=0 \\ n_{k-1}\not\equiv n_k \ (2)}}^{\infty} \frac{q^{\sum_{i,j=1}^k B_{ij}n_in_j + \frac{\lambda}{2}(n_{k-1}-n_k)}}{(q;q)_{n_1}\cdots(q;q)_{n_k}}$$

so that we have two fermionic representations for every character χ_λ.

Equating each of the fermionic forms with the corresponding bosonic form we obtain the following two q-series identities:

$$\sum_{\substack{n_1,\dots,n_k=0 \\ n_{k-1}+n_k\equiv \sigma \ (2)}}^{\infty} \frac{q^{\sum_{i,j=1}^k B_{ij}n_in_j + \frac{\lambda}{2}(n_{k-1}-n_k+\sigma) - \frac{1}{4}\sigma k}}{(q;q)_{n_1}\cdots(q;q)_{n_k}} \tag{4a}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{kn^2+(\lambda-\sigma k)n}$$

for $\lambda \in \{0, \dots, k\}$ and

$$\sum_{\substack{n_1,\dots,n_k=0\\ n_{k-1}+n_k\equiv\sigma\ (2)}}^{\infty} \frac{q^{\sum_{i,j=1}^k B_{ij}n_in_j+\sum_{i=k-\lambda}^{k-2}(i-k+\lambda+1)n_i+\frac{\lambda}{2}(n_{k-1}+n_k+\sigma)-\frac{1}{4}\sigma k}}{(q;q)_{n_1}\cdots(q;q)_{n_k}} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(2n-\sigma+1)q^{kn^2+(\lambda-\sigma k)n} \tag{4b}$$

for $\lambda \in \{1, \dots, k-1\}$. In both formulas σ is either zero or one.

We remark that by the Jacobi triple product identity [2] the right-hand side of (4a) may also be written in product form as

$$\frac{(-q^{k+\lambda-\sigma k}, -q^{k-\lambda+\sigma k}, q^{2k}; q^{2k})_\infty}{(q;q)_\infty},$$

where

$$(a, q/a, q; q)_\infty = \prod_{i=1}^{\infty}(1-aq^{i-1})(1-q^i/a)(1-q^i).$$

3. Proof of (4a) and (4b)

As a first step we rewrite (4a) and (4b) by replacing the summation variables n_{k-1} and n_k by n and m respectively. Also eliminating explicit reference to the inverse Cartan matrix B using (3), we get

$$\sum_{\substack{n,m=0\\ n+m\equiv\sigma\ (2)}}^{\infty} \frac{q^{\frac{k}{4}(n^2+m^2-\sigma)+\frac{k-2}{2}nm+\frac{\lambda}{2}(n-m+\sigma)}}{(q;q)_n(q;q)_m} \times \sum_{n_1,\dots,n_{k-2}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-2}^2+(n+m)(N_1+\cdots+N_{k-2})}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{kn^2+(\lambda-\sigma k)n} \tag{5a}$$

and

$$\sum_{\substack{n,m=0\\ n+m\equiv\sigma\ (2)}}^{\infty} \frac{q^{\frac{k}{4}(n^2+m^2-\sigma)+\frac{k-2}{2}nm+\frac{\lambda}{2}(n+m+\sigma)}}{(q;q)_n(q;q)_m} \times \sum_{n_1,\dots,n_{k-2}=0}^{\infty}\frac{q^{N_1^2+\cdots+N_{k-2}^2+N_{k-\lambda}+\cdots+N_{k-2}+(n+m)(N_1+\cdots+N_{k-2})}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-2}}} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(2n-\sigma+1)q^{kn^2+(\lambda-\sigma k)n}, \tag{5b}$$

where $N_i = n_i + n_{i+1} + \cdots + n_{k-2}$. We note that the quadratic form involving the N_i may alternatively be expressed in terms of the submatrix T of B given by

$T_{ij} = \min(i,j)$ for $1 \le i,j \le k-2$. Specifically,

$$N_1^2 + \cdots + N_{k-2}^2 = \sum_{i,j=1}^{k-2} T_{ij} n_i n_j.$$

Proof of (5a). To prove (5a) we denote its left-hand side by $L_{\lambda,k,\sigma}$. Shifting the summation index $n \to 2n-m-\sigma$ and replacing $k \to k+1$ we obtain

$$\begin{aligned}(6)\quad L_{\lambda,k+1,\sigma} = \sum_{n=0}^{\infty}\sum_{m=0}^{2n-\sigma} &\frac{q^{kn^2+(n-m)(n-m+\lambda-\sigma)-\sigma kn}}{(q;q)_{2n-m-\sigma}(q;q)_m} \\ &\times \sum_{n_1,\dots,n_{k-1}}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+(2n-\sigma)(N_1+\cdots+N_{k-1})}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}},\end{aligned}$$

where now

$$N_i = n_i + \cdots + n_{k-1}$$

and $\lambda \in \{0,\dots,k+1\}$.

In order the evaluate the multisum on the second line we consider the more general expression

$$(7)\qquad Q_{k,i}(x) = \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{x^{N_1+\cdots+N_{k-1}} q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}$$

for $i \in \{1,\dots,k\}$. This multisum was first introduced by Andrews [1, Equation (2.5)] in his proof of the analytic form of Gordon's partition identities. (In the notation of Andrews' book *Partition Theory* we have $Q_{k,i}(x) = J_{k,i}(0;x;q)$, see [2, Equation (7.3.8)].)

In [1, Equation (2.1)] (see also [2, Equations (7.2.1) & (7.2.2)]) we find the following single-sum form for $Q_{k,i}$:

$$Q_{k,i}(x) = \frac{1}{(xq;q)_\infty} \sum_{j=0}^{\infty} (-1)^j x^{kj} q^{\binom{j}{2}+kj^2+(k-i+1)j} \bigl(1 - x^i q^{i(2j+1)}\bigr) \frac{(xq;q)_j}{(q;q)_j}.$$

This in fact shows that $Q_{k,i}$ coincides with functions studied earlier by Rogers [17] and Selberg [18]. From the above we infer that

$$\begin{aligned}(8)\quad Q_{k,k-\lambda+1}(q^{2n-\sigma}) = \frac{1}{(q;q)_\infty} &\sum_{j=0}^{\infty} (-1)^j q^{\binom{j}{2}+kj^2+(\lambda-\sigma k)j+2knj} \\ &\times \bigl(1 - q^{(k-\lambda+1)(2j+2n-\sigma+1)}\bigr) \frac{(q;q)_{j+2n-\sigma}}{(q;q)_j}.\end{aligned}$$

Let us now return to (6). By (7) the multisum on the second line of (6) may be identified as $Q_{k,k}(q^{2n-\sigma})$ and by (8) with $\lambda = 1$ this may be simplified to a single sum. Therefore

$$\begin{aligned}L_{\lambda,k+1,\sigma} = \frac{1}{(q;q)_\infty} \sum_{j,n=0}^{\infty} \sum_{m=0}^{2n-\sigma} &(-1)^j q^{\binom{j+1}{2}+k(j+n)^2+(n-m)(n-m+\lambda-\sigma)-\sigma k(j+n)} \\ &\times \bigl(1 - q^{k(2j+2n-\sigma+1)}\bigr) \frac{(q;q)_{j+2n-\sigma}}{(q;q)_j (q;q)_m (q;q)_{2n-m-\sigma}}.\end{aligned}$$

Our next step is to shift the summation indices $n \to n-j$ and $m \to m-j$, resulting in

$$L_{\lambda,k+1,\sigma} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} q^{kn(n-\sigma)+(n-m)(n-m+\lambda-\sigma)} (1-q^{k(2n-\sigma+1)}) \\ \times \sum_{j=0}^{\min(m,2n-m-\sigma)} (-1)^j q^{\binom{j+1}{2}} \frac{(q;q)_{2n-j-\sigma}}{(q;q)_j (q;q)_{m-j} (q;q)_{2n-m-j-\sigma}}.$$

Employing standard basic hypergeometric notation [14] this may also we written as

$$L_{\lambda,k+1,\sigma} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} q^{kn(n-\sigma)+(n-m)(n-m+\lambda-\sigma)} (1-q^{k(2n-\sigma+1)}) \\ \times \frac{(q;q)_{2n-\sigma}}{(q;q)_m (q;q)_{2n-m-\sigma}} \, {}_2\phi_1 \left[\begin{matrix} q^{-m}, q^{-(2n-m-\sigma)} \\ q^{-(2n-\sigma)} \end{matrix} ; q, q \right]. \tag{9}$$

To proceed we need the q-Chu–Vandermonde sum [14, Equation (II.6)]

$${}_2\phi_1 \left[\begin{matrix} a, q^{-n} \\ c \end{matrix} ; q, q \right] = a^n \frac{(c/a;q)_n}{(c;q)_n}.$$

Hence the ${}_2\phi_1$ series may be summed to

$$\frac{(q;q)_m (q;q)_{2n-m-\sigma}}{(q;q)_{2n-\sigma}} \tag{10}$$

leading to

$$L_{\lambda,k+1,\sigma} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} q^{kn(n-\sigma)+(n-m)(n-m+\lambda-\sigma)} (1-q^{k(2n-\sigma+1)}).$$

The remainder of the proof requires only elementary manipulations:

$$\begin{aligned} L_{\lambda,k+1,\sigma} &= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^{n-\sigma} q^{kn(n-\sigma)+m(m-\lambda+\sigma)} (1-q^{k(2n-\sigma+1)}) \\ &= \frac{1}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} q^{m(m-\lambda+\sigma)} \sum_{n=\max(-m,m+\sigma)}^{\infty} \Bigl(q^{kn(n-\sigma)} - q^{k(n+1)(n+1-\sigma)} \Bigr) \\ &= \frac{1}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} q^{(k+1)m^2-(\lambda-\sigma(k+1))m}. \end{aligned}$$

Finally replacing $k \to k-1$ and changing the summation index $m \to -n$ completes the proof of (5a). $\square$

Proof of (5b). As in the proof of (5a) we denote the left-hand side of (5b) by $L_{\lambda,k,\sigma}$. Again we carry out the shift $n \to 2n-m-\sigma$ in the summation index, and replace $k \to k+1$. Thus

$$L_{\lambda,k+1,\sigma} = \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} \frac{q^{kn^2+(n-m)(n-m-\sigma)+(\lambda-\sigma k)n}}{(q;q)_{2n-m-\sigma} (q;q)_m}$$

$$\times \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{k-\lambda+1}+\cdots+N_{k-1}+(2n-\sigma)(N_1+\cdots+N_{k-1})}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}}.$$

From (7) we infer that the second line is $J_{k,k-\lambda+1}(q^{2n-\sigma})$ so that we may invoke (8) to obtain

$$L_{\lambda,k+1,\sigma} = \frac{1}{(q;q)_\infty} \sum_{j,n=0}^{\infty} \sum_{m=0}^{2n-\sigma} (-1)^j q^{\binom{j}{2}+k(j+n)^2+(\lambda-\sigma k)(j+n)+(n-m)(n-m-\sigma)} \\ \times (1-q^{(k-\lambda+1)(2j+2n-\sigma+1)}) \frac{(q;q)_{j+2n-\sigma}}{(q;q)_j(q;q)_m(q;q)_{2n-m-\sigma}}.$$

Following the earlier proof we shift $n \to n-j$ and $m \to m-j$, and use basic hypergeometric notation to find

$$L_{\lambda,k+1,\sigma} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} q^{kn^2+(\lambda-\sigma k)n+(n-m)(n-m-\sigma)}(1-q^{(k-\lambda+1)(2n-\sigma+1)}) \\ \times \frac{(q;q)_{2n-\sigma}}{(q;q)_m(q;q)_{2n-m-\sigma}} \, {}_2\phi_1\left[\begin{matrix} q^{-m}, q^{-(2n-m-\sigma)} \\ q^{-(2n-\sigma)} \end{matrix}; q, 1\right].$$

This time we need the second form of the q-Chu–Vandermonde sum [14, Equation (II.7)]

$$(11) \qquad {}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right] = \frac{(c/a;q)_n}{(c;q)_n}$$

to sum the ${}_2\phi_1$ series to

$$q^{(2n-\sigma)m-m^2} \frac{(q;q)_m(q;q)_{2n-m-\sigma}}{(q;q)_{2n-\sigma}}.$$

Hence

$$(12) \qquad \begin{aligned} L_{\lambda,k,\sigma} &= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} q^{kn^2+(\lambda-\sigma k)n}(1-q^{(k-\lambda)(2n-\sigma+1)}) \\ &= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (2n-\sigma+1) q^{kn^2+(\lambda-\sigma k)n}(1-q^{(k-\lambda)(2n+1)}) \\ &= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (2n-\sigma+1) q^{kn^2+(\lambda-\sigma k)n}, \end{aligned}$$

establishing (5b). □

4. Discussion

The $c_{k,1}$ character identities proved in this paper admit polynomial analogues. Defining the q-binomial coefficient as

$$(13) \qquad \begin{bmatrix} n+m \\ n \end{bmatrix} = \frac{(q;q)_{n+m}}{(q;q)_n(q;q)_m}$$

for n, m nonnegative integers, and assuming $k \geq 3$, we for example have
(14)
$$\sum_{\substack{n_1,\dots,n_k=0\\ n_{k-1}\equiv n_k\ (2)}}^{\infty} z^{\frac{1}{2}(n_{k-1}-n_k)} q^{\sum_{i,j=1}^k B_{ij}n_in_j} \prod_{i=1}^k \begin{bmatrix} n_i+m_i \\ n_i \end{bmatrix} = \sum_{n=-\infty}^{\infty} z^n q^{kn^2} \begin{bmatrix} 2L \\ L-kn \end{bmatrix}.$$

Here the m_i appearing in the q-binomial coefficients are fixed by

$$m_i = \sum_{j=1}^k B_{ij}(2L\delta_{j,1} - 2n_j).$$

When L tends to infinity and z is specialised to q^λ the identity (14) simplifies to (4a) with $\sigma = 0$. It is interesting to note that for $q = 1$ it provides an identity for the number of walks of length $2L$ on the rooted cyclic graph C_{2k} beginning and terminating at the root. Here the parameter z in the generating function serves to keep track of the number of cycles of the rooted walks on C_{2k}.

The previous method of proof fails to also deal with (14) but, as will be shown in Appendix A, (14) may be proved by induction on k.

Finally we remark that if we replace $q \to 1/q$ in (14) and then let L tend to infinity we obtain the dual identity

$$\begin{aligned}\sum_{\substack{m_1,\dots,m_k=0\\ m_1,\dots,m_{k-2}\equiv 0\ (2)\\ m_{k-1}\equiv m_k\ (2)}}^{\infty} &\frac{z^{\frac{1}{2}(m_{k-1}-m_k)} q^{\frac{1}{4}\sum_{i,j=1}^k C_{ij}m_im_j}}{(q;q)_{m_1}} \prod_{i=2}^k \begin{bmatrix} n_i+m_i \\ m_i \end{bmatrix} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{k(k-1)n^2} \\
&= \frac{(-zq^{k(k-1)}, -q^{k(k-1)}/z, q^{2k(k-1)}; q^{2k(k-1)})_\infty}{(q;q)_\infty},\end{aligned}$$

where $C = B^{-1}$ is the D_k Cartan matrix and

$$n_i = -\frac{1}{2}\sum_{j=1}^k C_{ij}m_j.$$

5. Postscript

Shortly after completing this paper B. Feigin, E. Feigin and Tipunin proved another family of character formulas for the $c_{k,1}$ models [7]. Replacing $p \to k$ and $s \to k - \lambda$ and $(n_+, n_-) \to (n, m)$ in [7, Theorem 1.1] the result of Feigin *et al.* reads

(15)
$$\begin{aligned}\chi^+_\lambda(q) + \chi^-_{k-\lambda}(q) &= q^{\varphi_\lambda} \sum_{n,m=0}^{\infty} \frac{q^{\frac{k}{4}(n+m)^2 + \frac{\lambda}{2}(n+m)}}{(q;q)_n(q;q)_m} \\
&\quad\times \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{k-\lambda}+\cdots+N_{k-1}+(n+m)(N_1+\cdots+N_{k-1})}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}},\end{aligned}$$

where

$$N_i = n_1 + \cdots + n_{k-1}.$$

When the sum over n and m is restricted to even (odd) values of $n+m$ we obtain $\chi^+_\lambda(q)$ $(\chi^-_{k-\lambda}(q))$, and in Appendix B the method used in Section 3 to prove the FGK conjectures is employed to establish that

$$\sum_{\substack{n,m=0\\ n+m\equiv\sigma\ (2)}}^{\infty} \frac{q^{\frac{k}{4}((n+m)^2-\sigma))+\frac{\lambda}{2}(n+m+\sigma)}}{(q;q)_n(q;q)_m} \times \sum_{n_1,\dots,n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_{k-\lambda}+\cdots+N_{k-1}+(n+m)(N_1+\cdots+N_{k-1})}}{(q;q)_{n_1}\cdots(q;q)_{n_{k-1}}} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(2n-\sigma+1)q^{kn^2+(\lambda-\sigma k)n}. \tag{16}$$

This is to be compared with (5b). Summing the above over $\sigma\in\{0,1\}$ yields (15).

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Appendix A.

In the appendix we prove (14). To begin we replace $n_k\to 2n+n_{k-1}$ on the left and $n\to -n$ on the right. Then equating coefficients of z^{-n} and finally replacing $k\to k+1$ yields

$$\sum_{n_1,\dots,n_k=0}^{\infty} q^{\sum_{i=1}^k N_i(N_i+2n)} \begin{bmatrix} L-(k-1)n-\sum_{i=1}^{k-1}N_i \\ n_k+2n\end{bmatrix}\begin{bmatrix} L-(k-1)n-\sum_{i=1}^{k-1}N_i \\ n_k\end{bmatrix} \times\prod_{i=1}^{k-1}\begin{bmatrix}2L-2in+n_i-2\sum_{j=1}^i N_j\\ n_i\end{bmatrix} = \begin{bmatrix}2L\\ L-(k+1)n\end{bmatrix},$$

where

$$N_i=n_1+\cdots+n_k.$$

Note that we may without loss of generality assume from now on that n is a nonnegative integer. Indeed, by the shift $n_k\to n_k-2n$ we obtain the same identity but with n replaced by $-n$.

Next we use the symmetry in n and m of the q-binomial coefficient (13) to rewrite the above multisum as

$$\sum_{n_1,\dots,n_k=0}^{\infty} q^{\sum_{i=1}^k N_i(N_i+2n)} \begin{bmatrix} L-(k-1)n-\sum_{i=1}^{k-1}N_i \\ L-(k+1)n-\sum_{i=1}^{k}N_i\end{bmatrix}\begin{bmatrix} L-(k-1)n-\sum_{i=1}^{k-1}N_i \\ n_k\end{bmatrix} \times\prod_{i=1}^{k-1}\begin{bmatrix}2L-2in+n_i-2\sum_{j=1}^i N_j\\ n_i\end{bmatrix} = \begin{bmatrix}2L\\ L-(k+1)n\end{bmatrix}. \tag{17}$$

At first sight this may not appear at all significant, but a close inspection reveals that we may now replace (13) by

$$\begin{bmatrix} n+m \\ n \end{bmatrix} = \begin{cases} \dfrac{(q^{m+1};q)_n}{(q;q)_n} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{cases} \tag{18}$$

The difference with the earlier definition is that the above q-binomial coefficient is non-zero when $n+m<0$ and $n \geq 0$. Clearly, if we can show that negative upper entries cannot occur in the q-binomial coeffients of (17), then the change of definition is justified. To achieve this we note that both q-binomial definitions imply that the summand of (17) vanishes unless $n_1, \ldots, n_k \geq 0$ and

$$\sum_{j=1}^{k} N_j \leq L-(k+1)n.$$

But this implies that

$$L-(k-1)n-\sum_{i=1}^{k-1} N_i \geq 2n \geq 0$$

and

$$2L-2in+n_i-2\sum_{j=1}^{i} N_j \geq 2(k-i+1)n+n_i \geq 0$$

as required.

We now proceed by proving the identity

$$\sum_{n_1,\ldots,n_k=0}^{\infty} q^{\sum_{i=1}^{k} N_i(N_i+m)} \begin{bmatrix} L_1+m-\sum_{i=1}^{k-1} N_i \\ L_1-\sum_{i=1}^{k} N_i \end{bmatrix} \begin{bmatrix} L_2-km-\sum_{i=1}^{k-1} N_i \\ n_k \end{bmatrix} \times \prod_{i=1}^{k-1} \begin{bmatrix} L_1+L_2-im+n_i-2\sum_{j=1}^{i} N_j \\ n_i \end{bmatrix} = \begin{bmatrix} L_1+L_2 \\ L_1 \end{bmatrix}, \tag{19}$$

where L_1, L_2, m are arbitrary integers. (For $L_1<0$ both sides trivially vanish since the sum over the n_i is bounded by $\sum_i N_i \leq L_1$.) The identity (17) is recovered by taking

$$(L_1, L_2, m) \to (L-(k+1)n, L+(k+1)n, 2n).$$

Before we continue let us remark that, generally, (19) is not true if one assumes definition (13) of the q-binomial coefficient.

Key to our proof of (19) are the polynomial form of the q-Pfaff–Saalschütz sum [2, Equation (3.3.11)]

$$\sum_{n=0}^{\min(b,d)} q^{n(n+a-b)} \begin{bmatrix} a \\ b-n \end{bmatrix} \begin{bmatrix} c \\ n \end{bmatrix} \begin{bmatrix} a+c+d-n \\ d-n \end{bmatrix} = \begin{bmatrix} a+d \\ b \end{bmatrix} \begin{bmatrix} a-b+c+d \\ d \end{bmatrix} \tag{20}$$

and its $d \to \infty$ limit (which corresponds to a polynomial analogue of the q-Chu–Vandermonde sum (11))

$$\sum_{n=0}^{b} q^{n(n+a-b)} \begin{bmatrix} a \\ b-n \end{bmatrix} \begin{bmatrix} c \\ n \end{bmatrix} = \begin{bmatrix} a+c \\ b \end{bmatrix}. \tag{21}$$

Thanks to (18) the above two summations are true for all integers a, b, c, d.

We now eliminate the variables n_i by $n_i = N_i - N_{i+1}$ (with $N_{k+1} = 0$) from (19) to obtain the equivalent formula

$$\sum_{N_1 \geq \cdots \geq N_k \geq 0} q^{\sum_{i=1}^k N_i(N_i+m)} \begin{bmatrix} L_1 + m - \sum_{i=1}^{k-1} N_i \\ L_1 - \sum_{i=1}^k N_i \end{bmatrix} \begin{bmatrix} L_2 - km - \sum_{i=1}^{k-1} N_i \\ N_k \end{bmatrix} \times \prod_{i=1}^{k-1} \begin{bmatrix} L_1 + L_2 - im + N_i - N_{i+1} - 2\sum_{j=1}^i N_j \\ N_i - N_{i+1} \end{bmatrix} = \begin{bmatrix} L_1 + L_2 \\ L_1 \end{bmatrix}. \tag{22}$$

For $k = 1$ this is

$$\sum_{N_1=0}^{L_1} q^{N_1(N_1+m)} \begin{bmatrix} L_1 + m \\ L_1 - N_1 \end{bmatrix} \begin{bmatrix} L_2 - m \\ N_1 \end{bmatrix} = \begin{bmatrix} L_1 + L_2 \\ L_1 \end{bmatrix},$$

which follows (21). Now assume that $k \geq 2$ and write the left-hand side of (22) as f_k. Then

$$f_k = \sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} q^{\sum_{i=1}^{k-1} N_i(N_i+m)} \prod_{i=1}^{k-2} \begin{bmatrix} L_1 + L_2 - im + N_i - N_{i+1} - 2\sum_{j=1}^i N_j \\ N_i - N_{i+1} \end{bmatrix}$$
$$\times \sum_{N_k \geq 0} q^{N_k(N_k+m)} \begin{bmatrix} L_1 + m - \sum_{i=1}^{k-1} N_i \\ L_1 - \sum_{i=1}^k N_i \end{bmatrix} \begin{bmatrix} L_2 - km - \sum_{i=1}^{k-1} N_i \\ N_k \end{bmatrix}$$
$$\times \begin{bmatrix} L_1 + L_2 - (k-1)m + N_{k-1} - N_k - 2\sum_{j=1}^{k-1} N_j \\ N_{k-1} - N_k \end{bmatrix}.$$

The sum over N_k may be performed by (20), resulting in

$$f_k = f_{k-1}.$$

A standard induction argument completes the proof.

Appendix B.

In this appendix we prove (16). First we shift $n \to 2n - m - \sigma$ to find

$$\mathrm{LHS}(16) = \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} \frac{q^{kn^2+(\lambda-\sigma k)n}}{(q;q)_{2n-m-\sigma}(q;q)_m} Q_{k,k-\lambda}(q^{2n-\sigma}).$$

with $Q_{k,i}(x)$ defined in (7). By (8) with $\lambda \to \lambda + 1$ this becomes

$$\mathrm{LHS}(16) = \frac{1}{(q;q)_\infty} \sum_{j,n=0}^{\infty} \sum_{m=0}^{2n-\sigma} (-1)^j q^{\binom{j+1}{2}+k(j+n)^2+(\lambda-\sigma k)(j+n)}$$
$$\times (1 - q^{(k-\lambda)(2j+2n-\sigma+1)}) \frac{(q;q)_{j+2n-\sigma}}{(q;q)_j (q;q)_m (q;q)_{2n-m-\sigma}}.$$

After the shifts $n \to n - j$ and $m \to m - j$ this is

$$\mathrm{LHS}(16) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} q^{kn^2+(\lambda-\sigma k)n}(1 - q^{(k-\lambda)(2n-\sigma+1)})$$
$$\times \frac{(q;q)_{2n-\sigma}}{(q;q)_m (q;q)_{2n-m-\sigma}} \, {}_2\phi_1 \left[\begin{matrix} q^{-m}, q^{-(2n-m-\sigma)} \\ q^{-(2n-\sigma)} \end{matrix} ; q, q \right]$$

$$
\begin{aligned}
&= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{2n-\sigma} q^{kn^2+(\lambda-\sigma k)n}\big(1-q^{(k-\lambda)(2n-\sigma+1)}\big) \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (2n-\sigma+1) q^{kn^2+(\lambda-\sigma k)n}.
\end{aligned}
$$

Here the second equality follows by noting that the same ${}_2\phi_1$ sum occurs in (9) so that it equates to (10). The last equality follows from (12).

References

1. G. E. Andrews, *An analytic generalization of the Rogers–Ramanujan identities for odd moduli*, Prod. Nat. Acad. Sci. USA **71** (1974), 4082–4085.
2. G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, Vol. 2, (Addison-Wesley, Reading, Massachusetts, 1976).
3. A. Berkovich, *Fermionic counting of RSOS-states and Virasoro character formulas for the unitary minimal series $M(\nu,\nu+1)$. Exact results*, Nucl. Phys. B **431** (1994), 315–348.
4. A. Berkovich and B. M. McCoy, *Continued fractions and fermionic representations for characters of $M(p,p')$ minimal models*, Lett. Math. Phys. **37** (1996), 49–66.
5. A. Berkovich, B. M. McCoy and A. Schilling, *Rogers–Schur–Ramanujan type identities for the $M(p,p')$ minimal models of conformal field theory*, Commun. Math. Phys. **191** (1998), 325–395.
6. S. Dasmahapatra, R. Kedem, T. R. Klassen, B. M. McCoy and E. Melzer, *Quasi-particles, conformal field theory, and q-series*, Internat. J. Modern Phys. B **7** (1993), 3617–3648.
7. B. Feigin, E. Feigin and I. Tipunin, *Fermionic formulas for $(1,p)$ logarithmic model characters in $\Phi_{2,1}$ quasiparticle realisation*, arXiv:hep-th/0704.2464.
8. M. A. I. Flohr, *On modular invariant partition functions of conformal field theories with logarithmic operators*, Internat. J. Modern Phys. A **11** (1996), 4147–4172.
9. M. Flohr, C. Grabov and M. Koehn, *Fermionic expressions for the characters of $c_{p,1}$ logarithmic conformal field theories*, arXiv:hep-th/0611241.
10. O. Foda and Y.-H. Quano, *Polynomial identities of the Rogers–Ramanujan type*, Internat. J. Modern Phys. A **10** (1995), 2291–2315.
11. O. Foda and Y.-H. Quano, *Virasoro character identities from the Andrews–Bailey construction*, Internat. J. Modern Phys. A **12** (1996), 1651–1675.
12. O. Foda, K. S. M. Lee and T. A. Welsh, *A Burge tree of Virasoro-type polynomial identities*, Internat. J. Modern Phys. A **13**, (1998), 4967–5012.
13. O. Foda, K. S. M. Lee, Y. Pugai and T. A. Welsh, *Path generating transforms*, in *q-Series from a Contemporary Perspective*, pp. 157–186, M. E. H. Ismail and D. W. Stanton eds., Contemp. Math. **254**, (AMS, Providence, 2000).
14. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, Vol. 35, second edition, (Cambridge University Press, Cambridge, 2004).
15. L. Lewin, *Dilogarithms and Associated Functions*, (Macdonald, London, 1958).
16. E. Melzer, *Fermionic character sums and the corner transfer matrix*, Int. J. Mod. Phys. A **9** (1994), 1115–1136.
17. L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. (2) **16** (1917), 315–336.
18. A. Selberg, *Über einige arithmetische Identitäten*, Avhl. Norske Vid. **8** (1936), 1–23.
19. S. O. Warnaar, *Fermionic solution of the Andrews–Baxter–Forrester model. I. Unification of CTM and TBA methods*, J. Stat. Phys. **82** (1996), 657–685.
20. S. O. Warnaar, *Fermionic solution of the Andrews–Baxter–Forrester model. II. Proof of Melzer's polynomial identities*, J. Stat. Phys. **84** (1996), 49–83.
21. S. O. Warnaar, *The Andrews–Gordon identities and q-multinomial coefficients*, Commun. Math. Phys. **184** (1997), 203–232.
22. T. A. Welsh, *Fermionic expressions for minimal model Virasoro characters*, Mem. Amer. Math. Soc. **175** (2005), no. 827, viii+160 pp.

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